A Conditional Short-Time Core-Growth Mechanism in an Explicit Forced Two-Vortex-Tube Model on the 3-Torus

Technical note draft
Prepared from exploratory analysis and model reduction work

Claim size and scope. This draft does not claim a solution to the Clay Navier–Stokes problem. It presents an explicit forced periodic model family, an organized reduction to a short-time threshold mechanism, one assembled conditional proposition, and numerical sanity checks supporting the mechanism. The precise late-time numerical threshold remains unsettled in the current formalized runs.

Abstract

This note develops a first-step mathematical framework for an explicit forced two-vortex-tube family on the three-dimensional torus. The model consists of smooth same-sign, nearly axis-aligned Gaussian vorticity tubes coupled to a smooth localized compressive-shear forcing field. The analysis localizes the dynamics to a core region and an adjacent shell, derives a localized enstrophy identity, isolates cutoff transport and diffusion terms, and shows how shell contributions may be bounded by shell enstrophy. A conditional short-time core-growth proposition is then assembled: if shell closure, coherence, coefficient bounds, projection-corrected forcing alignment, and positive axial strain hold on a controlled interval, then the core enstrophy satisfies a lower differential inequality with explicit gain and loss terms. Sanity checks indicate that the mechanism is robust to moderate cutoff changes and restart-time shifts, and that Leray projection weakens but does not reverse the core-positive forcing strain. Matched long-hold tests in the current formalized implementation support a mechanism paper but do not yet support a sharp numerical threshold claim at the previously suspected forcing band. The shell appears to act not only as bookkeeping but also as a late-time regulator or reservoir in the current model family. The result is an organized theorem lane for a first paper, together with a clear inventory of the remaining unresolved constants.

1. Purpose and position of the draft

The immediate goal is not to solve Navier–Stokes globally. The goal is to make one explicit model family mathematically legible, reduce the threshold story to named terms, and identify exactly where the real unresolved hinge lives.

- The model is periodic, forced, and explicit.
- The proposition is short-time and conditional.
- The numerics are used as mechanism support and sanity checks, not as proof.
- The paper is written for an educated technical reader rather than a general audience.

2. Fixed model family

Work on the three-dimensional flat torus $T^3 = [0, 2\pi)^3$ with viscosity $\nu > 0$. The velocity field $u(x,t)$ and pressure $p(x,t)$ satisfy:

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu\Delta u + A(t)F(x), \quad \nabla \cdot u = 0, \quad u(x,0) = u_0(x).$$

The forcing amplitude uses a ramp-then-freeze law:

$$A(t) = A_{max}\, t / T_r \quad \text{for } 0 \le t \le T_r, \qquad A(t) = A_{max} \quad \text{for } t > T_r.$$

The initial vorticity consists of two slightly asymmetric same-sign Gaussian tubes aligned with the e_3 axis:

$$\omega_0(x) = (0, 0, \Gamma_1 \exp(-r_1(x)^2 / \sigma_1^2) + \Gamma_2 \exp(-r_2(x)^2 / \sigma_2^2)).$$

The initial velocity u_0 is the unique zero-mean divergence-free field on T^3 with curl $u_0 = \omega_0$.

3. Core/shell localization

Choose a smooth core cutoff χ and a shell cutoff ψ with $\psi \equiv 1$ on the transition shell of χ. Define:

$$E_\chi(t) = \int_{T^3} \chi(x)\, |\omega(x,t)|^2\, dx, \qquad E_\psi(t) = \int_{T^3} \psi(x)\, |\omega(x,t)|^2\, dx.$$

The working interpretation is that E_χ measures core enstrophy in the tube interaction region, while E_ψ captures enstrophy in the shell where cutoff transport and diffusion-correction terms live.

4. Localized enstrophy identity

With ω = curl u, the vorticity equation is:

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu\Delta\omega + A(t) \operatorname{curl} F.$$

Multiplying by $\chi\omega$ and integrating gives the localized core balance:

$$1/2\, d/dt\, E_\chi = -\nu D_\chi + S_\chi + A(t)F_\chi + T_\chi + C_\chi,$$

where D_χ is localized diffusion, S_χ is localized stretching, F_χ is the localized forcing-curl term, T_χ is transport across the cutoff, and C_χ is the diffusion-correction term arising when derivatives hit χ.

This identity is the paper's bookkeeping backbone. It separates the genuinely dangerous mechanism from the localization taxes.

5. Trash terms and shell control

The transport and diffusion-correction terms satisfy shell-weighted bounds of the form:

$$|T_\chi| + |C_\chi| \le \left(1/2\, ||\nabla\chi||_\infty\, ||u||_{L^\infty(\Sigma_\chi)} + (\nu/2)||\Delta\chi||_\infty \right) E_\psi(t).$$

Thus the cutoff terms are not free-floating threats. They are taxes paid through shell enstrophy. Applying the same localized identity to ψ yields a shell differential inequality:

$$d/dt\, E_\psi(t) \le K_\psi(t)\, E_\psi(t) + B_\psi(t).$$

A Grönwall argument gives an explicit upper bound $U_\psi(t)$. In parallel, the core identity yields a lower bound $L_\chi(t)$ for $E_\chi(t)$ on short intervals, provided diffusion and coefficient terms remain bounded. If $U_\psi(T) \leq \eta L_\chi(T)$, then shell closure follows on [0,T]:

$$E_\psi(t) \leq \eta E_\chi(t) \quad \text{for all } 0 \leq t \leq T.$$

In the proof lane this makes the shell a controlled loss layer. In the diagnostics, however, the shell also appears to play a genuine late-time regulatory role.

6. Short-time coefficient and persistence bounds

Smooth local existence gives short-time Sobolev control. For any T below the local smoothness horizon, define

$$U_2(T) = \sup_{0\leq t\leq T} ||u(t)||_{H^2}, \quad U_3(T) = \sup_{0\leq t\leq T} ||u(t)||_{H^3}.$$

Sobolev embedding on T^3 then yields short-time finite bounds for $||u||_{L^\infty}$ and $||\nabla u||_{L^\infty}$, which in turn produce a coefficient bound $M_0(T)$ for the core lower inequality. A corrected persistence estimate, keeping the diffusion term honestly as a bounded tax rather than dropping it, gives

$$E_\chi(t) \geq (E_\chi(0) + R_0/M_0) \exp(-2M_0 t) - R_0/M_0, \quad R_0 = \nu D_0 + Q_0.$$

This proves only short-time persistence, not growth. Its role is simpler: the core survives long enough for the production mechanism to matter.

7. Production mechanism and short-time threshold proposition

Define the net core production term

$$P_\chi(t) = S_\chi(t) + A(t)F_\chi(t).$$

The central reduction of the paper is that if one can prove a lower bound

$$P_\chi(t) \geq \mu E_\chi(t) - P_0,$$

then the core obeys a lower differential inequality with explicit gain and loss terms. A decomposition of ω into axial and misaligned parts, together with sign coherence and positive axial strain, leads to a candidate stretching coefficient

$$\mu_S = (1 - \delta^2) \lambda_* - (2\delta + \delta^2) G_0.$$

The curl-based forcing contribution contributes an additional aligned production term

$$A(t)F_\chi(t) \geq A(t) (c_F / \Omega_0) E_\chi(t).$$

Assembled together, the effective frozen-phase production-minus-loss coefficient is

$$a_{frz} = (1 - \delta^2) \lambda_* - (2\delta + \delta^2) G_0 + A_{max} c_F / \Omega_0 - \eta C_0.$$

If $a_{frz} > 0$ and the core lies above the associated barrier level, then the lower comparison forces core growth on the interval of validity. This is the paper's main assembled conditional proposition.

8. Forcing-side constants and the projection defect

The raw forcing field has the form $G(x) = \varphi(x)V(x)$, where φ is a Gaussian envelope centered at the tube interaction region and V is a compressive-shear vector field. Two forcing-side quantities matter most:

- c_F, the forcing-core alignment coefficient in the vorticity production term;
- m_F, the projected axial-strain lower bound for $F = P G$.

A useful simplification is that curl F = curl G, so the forcing-core alignment constant c_F is fully explicit and does not depend on the Leray projection defect. The projected axial strain, however, does depend on the scalar correction q in $F = G - \nabla q$. The key positivity condition is

$$m_F \geq m_G - \|\partial_{33} q\|_{L^\infty(\mathrm{supp}\,\chi)}.$$

Thus the projection defect is one of the paper's explicit remaining teeth: the raw field is axially extensional in the core, and the question is whether projection weakens that strain without reversing its sign.

9. Restarted Stokes reference and perturbation control

To isolate axial-strain production in the frozen phase, restart the analysis at a reference time t_0 and define U as the linear Stokes reference with $U(t_0) = u(t_0)$. The nonlinear remainder $v = u - U$ then satisfies $v(t_0) = 0$.

A Duhamel formula and Sobolev control yield the perturbation bound

$$\|\partial_3 v_3(t)\|_{L^\infty(\mathrm{supp}\,\chi)} \leq C_* (t - t_0) U_4(t_0, \tau)^2.$$

A separate lower bound for the linear/reference strain gives

$$\partial_3 U_3(x,t) \geq m_{init} - C_H \nu (t - t_0) H_{init} + \int_{t_0}^t A(s) \left(m_F - C_H \nu (t - s) M_F^{(2)} \right) ds.$$

Combining the two yields positivity of the full axial strain on a short interval whenever the forcing-driven slope beats both the linear smoothing penalty and the nonlinear perturbation penalty.

10. Interpreting the threshold constants

The threshold formula is ugly, but it is interpretable. The constants split naturally into three groups:

- Mechanism constants: δ, λ_*, c_F. These determine whether the core is aligned, stretched, and supported in the right direction.
- Localization taxes: η and C_0. These record shell pollution and cutoff overhead.
- Regularity/control constants: G_0, Ω_0, D_0, $U_2(T)$, $U_3(T)$, $U_4(t_0,\tau)$, and P_0. These are solution-side short-time control quantities rather than geometric design parameters.

This makes the threshold story readable to an educated technical reader: growth is favored by strong aligned axial strain and forcing support, and disfavored by misalignment, shell losses, and regularity taxes.

11. Numerical diagnostics and sanity checks

The numerical role of the diagnostics is not proof but mechanism support. The following checks were run on the formalized model family.

11.1 Cutoff robustness

Using tighter, baseline, and wider core/shell pairs, the sign of the short-window core-growth story did not flip. Late core-enstrophy slopes remained positive in the short-window test, and mean core axial strain stayed positive across moderate cutoff variations. This suggests the mechanism is not an artifact of one delicate cutoff placement. However, the pointwise minimum axial strain remained negative, so the strongest pointwise positivity story is not supported by this check alone.

11.2 Restart-time robustness

Restarting the linear/reference split slightly before freeze, at freeze, and slightly after freeze preserved the sign of the mechanism. Across these restart choices, mean core axial strain remained positive in the short analysis window and the core-enstrophy slope remained positive. This is strong evidence that the short-time mechanism is not merely a restart-frame artifact.

11.3 Projection-defect sanity

In the core, the raw axial forcing strain ∂_3 G_3 was everywhere positive, and the projected strain ∂_3 F_3 remained everywhere positive as well. Projection weakened the mean core-positive strain substantially but did not reverse its sign in the tested configuration. This supports the forcing-side argument that the projection defect is a weakening term rather than a sign-killing one.

11.4 Long-hold threshold repair tests

Matched longer-hold tests in the formalized model changed the earlier threshold narrative. In the current longer-hold formalized runs, A = 5.0 turned over late, and the matched A = 5.25 run also turned over late. That means the sharp numerical threshold at 5 versus 5.25 is not currently robust in the formalized implementation, even though higher forcing still pushes the peak higher and delays decay.

11.5 Shell as regulator rather than mere trash

A more interesting late-time pattern emerged when tracking the shell/core ratio E_ψ / E_χ along with core growth. In the matched turnover runs, the ratio dropped while the core was organizing and growing, bottomed out before the core peak, and then rose again while the core strain weakened and turnover followed. This suggests the shell is not only bookkeeping in the short-time proof lane. It may also act as a late-time regulator or reservoir in the full dynamics.

11.6 Full-versus-lean comparison

A matched full-versus-lean long-hold comparison for A = 5.0 produced very similar peak times, late slopes, shell/core ratios, and late mean core axial strain. This suggests that the threshold blur in the formalized model is not primarily a full-versus-lean implementation artifact.

12. What is actually proved versus still conditional

What the current draft legitimately establishes:

- An explicit forced two-vortex-tube model family on T^3.

- A localized core/shell formulation with a precise enstrophy balance.
- Reduction of cutoff taxes to shell-energy control.
- A conditional short-time core-growth proposition with explicit gain and loss coefficients.
- A clear forcing-side decomposition into alignment and projection-defect pieces.
- Numerical sanity checks showing that the mechanism survives several basic robustness tests.

What is not yet established:

- A sharp numerical threshold at the originally suspected 5 to 5.25 forcing band in the formalized long-hold model.
- An unconditional theorem closing every constant without auxiliary hypotheses.
- A long-time or global result.
- Anything close to a solution of the Clay Navier–Stokes problem.

13. Recommended claim for a first public paper

The right claim size for a first paper is: "Here is an explicit forced periodic two-vortex-tube model, an organized short-time threshold mechanism, an assembled conditional proposition, and numerical evidence that the shell plays a real late-time regulatory role." That claim is smaller than a solution and stronger than a loose numerical note. It is also honest.

14. Open technical teeth

The main unresolved teeth are now sharply identified:

- Sharper projection-defect control in the estimate for m_F.
- A cleaner bridge from the explicit forcing-side positivity to the actual solution-side axial-strain constant λ_*.
- Better understanding of the shell as a regulator rather than treating it only as lower-order bookkeeping.
- A decision on whether the first paper should stop at the conditional proposition or push one layer farther toward a genuinely closed theorem for a subclass.

15. Closing note

The point of this draft is not to overclaim. Its purpose is to make the mechanism explicit, to identify the precise locations of the remaining difficulty, and to hand the next stage to someone with the time and mathematical specialization to sharpen the unresolved constants. As a first step, this is already enough for a serious technical note.

Appendix A. Minimal notation map

Symbol	Meaning
E_χ	Core-localized enstrophy
E_ψ	Shell-localized enstrophy
D_χ	Localized diffusion tax
S_χ	Localized stretching term
F_χ	Localized forcing-curl contribution
T_χ, C_χ	Cutoff transport and cutoff diffusion-correction terms
δ	Coherence defect (misalignment size)
λ_*	Positive full axial-strain lower bound in the core
c_F	Forcing-core alignment coefficient
ηC_0	Effective shell/cutoff loss term
Ω_0	Short-time vorticity cap used in the forcing lower bound
m_F	Projected forcing axial-strain lower bound

Draft status: mechanism paper skeleton; not a Clay-solution claim.